VSMCOLLEGEOFENGINEERING

AUTONOMOUS

AccreditedbyNAACwith'A'Grade-3.23/4.00CGPA

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Department of

ELECTRICAL ELECTRONIC ENGINEERING

NUMERICAL METHODS AND COMPLEX VARIABLES

SUBJECTMATERIAL

YEAR:II SEMESTER: I

Regulaton:VR23 **SubjectCode:VR2321003**

Preparedby

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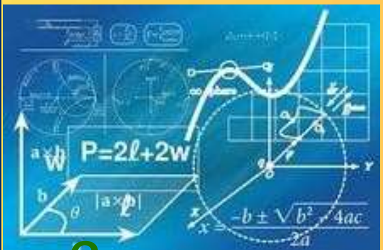
MATHS Department



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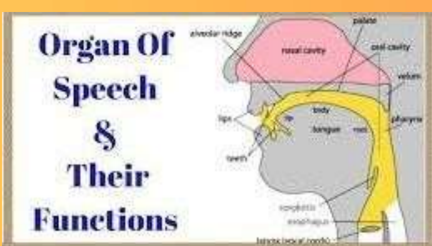

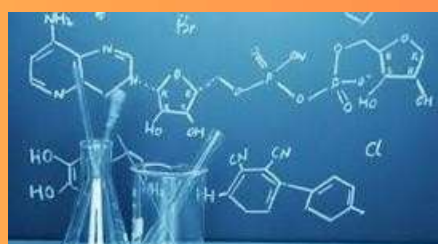

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Department of electrical electronic engineering

Subject Material

NUMERICAL METHODS AND COMPLEX VARIABLES

(only for Electrical electronic Engineering)

II B.TECH ISEM

Regulation: VR23
SubjectCode:VR2321003

VSM COLLEGEOFENGINEERING
Ramachandrapuram-533255

COMPLEX VARIABLES & NUMERICAL METHODS

Course Objectives:

- To elucidate the different numerical methods to solve nonlinear algebraic equations.
- To disseminate the use of different numerical techniques for carrying out numerical integration.
- To familiarize the complex variables
- To equip the students to solve application problems in their disciplines.

Course Outcomes:

1. Evaluate the approximate roots of polynomial and transcendental equations by different algorithms. Apply newton's forward& backword interpolation and Lagrange's formulae for equal and unequal intervals (L3)
2. Apply numerical integral techniques to different engineering problems .Apply different algorithms for approximating the solutions of ordinary differential equations with initial conditions to its analytical computations (L3)
3. Apply Cauchy-Riemann equations to complex functions in order to determine whether a given continuous function is analytic (L3)
4. Evaluate the Taylor and Laurent expansions of simple functions, determining the nature of the singularities and calculating residu theorem make use of the cauchy residue theorem to evaluate certain integrals (L3)
5. Explain properties of various types of conformal mapping

UNIT – I:

Iterative Methods:

Introduction – Solutions of algebraic and transcendental equations: Bisection method – Secant method – Method of false position – General Iteration method – Newton-Raphson method (Simultaneous Equations)

Interpolation: Newton's forward and backward formulae for interpolation – with unequal intervals – Lagrange's interpolation formula

UNIT – II:

Numerical integration, Solution of ordinary differential equations with initial conditions:

Trapezoidal rule– Simpson's $1/3^{rd}$ and $3/8^{th}$ rule– Solution of initial value problems by Taylor's series– Picard's method of successive approximations– Euler's method –Runge- Kutta method (second and fourth order) – Milne's Predictor and Corrector Method

UNIT – III:

Functions of a complex variable and Complex integration:

Introduction – Continuity – Differentiability – Analyticity –Cauchy-Riemann equations in Cartesian and polar coordinates – Harmonicand conjugate harmonic functions – Milne – Thompson method.

Complex integration: Line integral – Cauchy's integral theorem – Cauchy's integral formula – Generalized integral formula (all without proofs) and problems on above theorems.

UNIT – IV:

Series expansions and Residue Theorem:

Radius of convergence – Expansion of function in Taylor's series, Maclaurin's series andLaurent series.

Types of Singularities: Isolated – Essentialsingularities –Pole of order m– Residues – Residue

theorem(without proof) – Evaluation of real integral of the types ? $f(x)dx$ and f $(\cos\theta, \sin\theta) d\theta$

UNIT – V:

Conformal mapping:

Transformation by e^z, lnz, z^2, z^n (n positive integer), Sin z, cos z, z + a/z. Translation, rotation, inversion and bilinear transformation – fixed point – cross ratio – properties – invariance of circles and cross ratio – determination of bilinear transformation mapping 3 given points .

Unit-1

UNIT-I

Iteration methods

Soultions of algebric and transidental equations:-

1, **Bisection method:-**

Bisection method is a simple etaration method to solve an equation suppose in now on the equation of the form of $y = f(x)$ as extraly one real rode b/w two real Numbers a and b.

* $f(a) \to -ve \ (<0)$ $f(a) \to +ve \ (>0)$
  $f(b) \to +ve \ (>0)$ $f(b) \to -ve \ (<0)$

* Let us bisect the interval (a,b) into two half intervals and find mid point. $x_0 = \frac{a+b}{2}$, $f(x_0)=0$, then x_1 is a root

* Repeat the process untill the root get approximatly same.

Sum:-

1, Find the real root of equation x^4-x-10 by using bisection method.

Sol:- $f(x) = x^4 - x - 10$

$f(0) = (0)^4 - 0 - 10 = -10 \ (<0)$

$f(1) = (1)^4 - 1 - 10 = 1 - 1 - 10 = -10 \ (<0)$

$f(2) = (2)^4 - 2 - 10 = 16 - 2 - 10 = 4 > 0$

then the real roots are $[1, 2]$

$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5 \ (+ve)$

$\Rightarrow F(1.5) = (1.5)^4 - 1.5 - 10$

$= -6.44 \ (-ve)$

Then the real roots are $[1.5, 2]$

$x_2 = \frac{1.5+2}{2} = 1.75$

$f(1.75) = (1.75)^4 - 1.75 - 10$

$= -2.371 \ (-ve)$

then the roots are $(1.75, 2)$

$x_3 = \frac{1.75+2}{2} = 1.87$

$f(1.87) = (1.87)^4 - 1.87 - 10$

$= 0.35 \ (+ve)$

then the roots are $(1.87, 1.75)$

$x_4 = \frac{1.87+1.75}{2} = 1.81$

$f(1.81) = (1.81)^4 - 1.81 - 10$
$= -1.07$ (-ve)

then the ~~real~~ roots are $(1.81, 1.87)$

$x_5 = \frac{1.81 + 1.87}{2} = 1.84$

$f(1.84) = (1.84)^4 - 1.84 - 10 =$

then the 5^{th} and 4^{th} real roots are approximately same!

2, Find the real root of the equation.

$f(x) = x^3 - x - 1$

$f(x) = x^3 - x - 1$

$f(0) = 0^3 - 0 - 1 = -1$ (<0)

$f(1) = (1)^3 - 1 - 1 = -1$ (<0)

$f(2) = (2)^3 - 2 - 1 = 5$ (>0)

then the real roots are $[1, 2]$

$x_1 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$ [+ve]

$\Rightarrow F[1.5] = (1.5)^3 - 1.5 - 1 = 0.87$ (+ve]

then the roots are $[1.5, 1]$

$x_2 = \frac{1.5+1}{2} = 1.25$

$f(1.25) = (1.25)^3 - 1.25 - 1 = (-0.47)$ (-ve)

then roots is [1.25, 1.5]

$x_3 = \frac{1.25+1.5}{2} = 1.375$

$f(1.375) = (1.37)^3 - 1.37 - 1 = (+0.22)$ (+ve)

the roots is [1.375, 1.5]

$x_4 = \frac{1.37+1.5}{2} = \frac{2.8}{2} = 1.4$

$f(1.4) = (1.4)^3 - 1.4 - 1 = 0.3$ (+ve)

then roots is (1.4, 1.3)

$x_5 = \frac{1.4+1.3}{2} = \frac{2.7}{2} = 1.35$

then roots is [1.375, 1.25]

$x_4 = \frac{1.375+1.25}{2} = \frac{2.375}{2} = 1.3125$ (+ve)

$f(1.1875) = (1.1875)^3 - 1.1875 - 1 =$

$= 1.674 - 1.1875 - 1 = -0.51$ (-ve)

then the 3rd and 4th real roots are approximately Same.

3. Find the real root of the equation.

$f(x) = x^3 + x - 1$

$f(x) = x^3 + x - 1$

$f(0) = 0^3 + 0 - 1 = -1 \; (<0)$

$f(1) = 1^3 + 1 - 1 = 1 \; (>0)$

then the roots is $(1, 0)$

$x_1 = \frac{1+0}{2} = \frac{1}{2} = 0.5$

$f(0.5) = (0.5)^3 + 0.5 - 1 = -0.375 \; (-ve)$

then the roots is $(0.5, 1)$

$x_2 = \frac{0.5+1}{2} = 0.75$

$f(0.75) = (0.75)^3 + (0.75) - 1 = 0.17 \; (+ve)$

then the roots are $(0.75; 0.5)$

$x_3 = \frac{0.75+0.5}{2} = 0.625$

$f(0.625) = (0.625)^3 + 0.625 - 1$

$= -0.13 \; (-ve)$.

Then the roots are $(0.625, 0.75)$

$$x_4 = \frac{0.625 + 0.75}{2} = 0.68$$

then the 3rd and 4th real roots are approximately same.

4, $x^3 - 3x - 5 = 0$

$f(x) = x^3 - 3x - 5 =$

$f(0) = 0^3 - 3(0) - 5 = -5 \ (<0)$

$f(1) = 1^3 - 3(1) - 5 = -7 \ (<0)$

$f(2) = 2^3 - 3(2) - 5 =$

$8 - 6 - 5 = -3 \ (<0)$

$f(3) = 3^3 - 3(3) - 5 = 13 \ (>0)$

then roots are $(3, 2)$

$$x_1 = \frac{3+2}{2} = \frac{5}{2} = 2.5$$

$f(2.5) = (2.5)^3 - 3(2.5) - 5$

$= 15.625 - 7.5 - 5 = 3.125 \ (+ve)$

then roots are $(2.5, 2)$

$$x_2 = \frac{2.5 + 2}{2} = \frac{4.5}{2} = 2.25$$

$f(2.25) = (2.25)^3 - 3(2.25) - 5$

$= 11.390625 - 6.75 - 5$

$= -0.359 \ (-ve)$

then the roots are $(2.25, 2.5)$

$x_3 = \frac{2.25 + 2.5}{2} = \frac{4.75}{2} = 2.375$

$f(2.375) = (2.375)^3 - 3(2.375) - 5$

$= 13.396484375 - 7.125 - 5$

$= 1.27 \ (+ve)$

then the roots are $(2.375, 2.25)$

$x_4 = \frac{2.375 + 2.25}{2} = 2.3125$

$x_4 = 2.3125 \ (+ve)$

then the 3rd and 4th real roots are approximatily Same.

5. $x^3 - 2x - 5 = 0$

Secant method:-

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

1, A real root of the equation $x^3 - 5x + 1 = 0$ lies in the interval [0, 1] perform four iteration by Secant method.

Sol:- Let $f(x) = x^3 - 5x + 1 = 0$

$f(0) = 0^3 - 5(0) + 1 = 1 \ (>0)$ (x_0)

$f(1) = 1 - 5(1) + 1 = -3 \ (<0)$ (x_1)

$x_0 = 0 \quad x_1 = 1$

$f(x_0) = 1 \quad f(x_1) = -3$

By using Secant method:

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

put $n = 1$

$$x_{1+1} = \frac{x_{1-1} f[x_1] - x_1 f(x_{1-1})}{f[x_1] - f[x_{1-1}]}$$

$$x_2 = \frac{x_0 f[x_1] - x_1 f[x_0]}{f(x_1) - f(x_0)}$$

$$= \frac{0(-3) - 1(1)}{-3-1} = \frac{-1}{-4} = \frac{1}{4} = 0.25$$

$$f(x_2) = f(0.25) = (0.25)^3 - 5(0.25) + 1$$

$$= -0.23 \quad (<0)$$

By using secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

put $n=2$

$$x_{2+1} = \frac{x_{2-1} f(x_2) - x_2 f(x_{2-1})}{f(x_2) - f(x_{2-1})}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_3 = \frac{1(-0.23) - 0.25(-3)}{-0.23 - (-3)}$$

$$x_3 = \frac{-0.23 + 0.75}{2.77} = \frac{0.52}{2.77} = 0.18$$

$$f(x_3) = f(0.18) = [0.18]^3 - 5[0.18] + 1$$

$$= 0.10$$

put $n=3$

$$x_{3+1} = \frac{x_{3-1} f(x_3) - x_3 f(x_{3-1})}{f(x_3) - f(x_{3-1})}$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0.25\,(0.10) - 0.18\,(-0.23)}{0.10 + 0.23}$$

$$= 0.201$$

$$f(x_4) = f[0.20] = (0.20]^3 - 5[0.20) + 1$$

$$= 0.0081 - 1.005 + 1 = 0.0031$$

put $n=4$

$$x_{4+1} = \frac{x_{4-1} f(x_4) - x_4 f(x_{4-1})}{f(x_4) - f(x_{4-1})}$$

$$= \frac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)}$$

$$= \frac{0.18\,(0.0031) - 0.201\,(0.10)}{(0.0031) - (0.10)}$$

$$= \frac{0.000558 - 0.0201}{-0.0969}$$

$$= \frac{-0.019542}{-0.0969}$$

$$= 0.201.$$

$x^3 - x - 4 = 0$

let $f(x) = x^3 - x - 4 = 0$

$f(0) = 0^3 - 0 - 4 = -4 \ (<0)$

$f(1) = 1^3 - 1 - 4 = -4 \ (<0)$

$f(2) = 2^3 - 2 - 4 = 8 - 2 - 4 = 2 \ (>0)$

$x_0 = 1 \qquad x_1 = 2$

$f(x_0) = 4 \qquad f(x_1) = 2$

By using secant method

$$x_{n+1} = \frac{x_{n-1} (f(x_n)) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

put $n = 1$

$$x_{1+1} = \frac{x_{1-1} f(x_1) - x_1 f(x_{1-1})}{f(x_1) - f(x_{1-1})}$$

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(2) - 2(-4)}{2+4} = \frac{2+8}{6} = \frac{10}{6} = 1.66$$

$f(x_2) = f(1.6) = (1.66)^3 - 1.66 - 4$

$= -1.08 \ (<0)$

put $n = 2$

$$x_{2+1} = \frac{x_{2-1} f(x_2) - x_2 f(x_{2-1})}{f(x_2) - f(x_{2-1})}$$

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{2(-1.08) - 1.66(2)}{-1.08 - 2}$$

$$= \frac{-2.16 - 3.32}{-3.08}$$

$$= \frac{-5.48}{-3.08} = 1.78$$

$$f(x_3) = f(1.77) = (1.78)^3 - 1.78 - 4$$
$$= -0.14$$

put $n = 3$

$$x_{3+1} = \frac{x_{3-1} f(x_3) - x_3 f(x_{3-1})}{f(x_3) - f(x_{3-1})}$$

$$x_4 = \frac{x_2 f(x_3) - 1.78(-1.08)}{-0.14 + 1.08}$$

$$= \frac{0.23 + 1.92}{0.94}$$

$$= 1.79$$

* $x^3 + x - 1 = 0$

$$f(x) = x^3 + x - 1 = 0$$
$$f(0) = 0^3 + 0 - 1 = -1 \ (<0)$$
$$f(1) = 1^3 + 1 - 1 = 1 \ (>0)$$

$x_0 = 0 \qquad x_1 = 1$

$f(x_0) = -1 \qquad f(x_1) = 1$

by using secant method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

put $n=1$

$$x_{1+1} = \frac{x_{1-1} f(x_1) - x_1 f(x_{1-1})}{f(x_1) - f(x_{1-1})}$$

$$x_2 = \frac{x_0 f(x_1) - x f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0(1) - 1(-1)}{1+1} = \frac{1}{2} = 0.5$$

$f(x_2) = f(0.5) = (0.5)^3 + 0.5 - 1$

$= -0.37$

put $n = 2$

$$x_{2+1} = \frac{x_{2-1} f(x_2) - x_2 f(x_{2-1})}{f(x_2) - f(x_{2-1})}$$

$$x_3 = \frac{x_1 f(x_1) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1(-0.37) - 0.5(1)}{-0.37 - 1}$$

$$= \frac{-0.87}{-1.37} = 0.63$$

$f(x_3) = f(0.63) = (0.63)^3 + (0.63) - 1$
$= -0.11$

Put $n = 3$

$$x_{3+1} = \frac{x_{3-1} f(x_3) - x_3 f(x_{3-1})}{f(x_3) - f(x_{3-1})}$$

$$x_4 = \frac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)}$$

$$= \frac{0.5(-0.11) - 0.63(-0.37)}{-0.11 + 0.37}$$

$$= \frac{-0.05 + 0.23}{0.251}$$

$$= \frac{0.1781}{0.26} \quad x_4 = 0.68$$

Regular - falsi method (or) (methods of falsi position)

* Regular falsi method is one of the simplest and most reliable numerical method to find an approximate root of a given capation $f(x) = 0$
* In this method, we first find two numbers 'a' and 'b' $(a < b)$ such that $f(a)$ and $f(b)$ are of opposite signs.

* This means that the root of the equation lie in (a,b)

* Then the next approximation is obtained by

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

1) solve $x^3 - 4x + 1 = 0$ by regular-falsi method upto 3-decimal places.

solu:- let $f(x) = x^3 - 4x + 1 = 0$

$$f(0) = 0^3 - 4(0) + 1 = 1 \ (>0)$$

$$f(1) = (1)^3 - 4(1) + 1 = -2 \ (<0)$$

Then the roots lies b/w $[0,1]$

$$a_0 = 0 \qquad b_0 = 1$$

$$f(a_0) = 1 \qquad f(b_0) = -2$$

by using Regular-falsi method.

$$a_1 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$

$$a_1 = \frac{0(-2) - 1(1)}{-2 - 1} = \frac{-1}{-3} = 1/3 = 0.333$$

$$f(a_1) = f(0.333) = (0.333)^3 - 4(0.333) + 1$$

$$= -0.295.$$

then the roots lies b/w [0.333, 0]

$a_1 = 0.333 \quad b_1 = 0$

$f(a_1) = -0.295 \quad f(b_1) = 1$

$$a_2 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{0.333 \times 1 - 0 \times (-0.295)}{1 - (-0.295)}$$

$$= \frac{0.333}{1.295} = 0.257$$

$$f(a_2) = f(0.257) = (0.257)^3 - 4(0.257) + 1$$

$$= -0.011$$

then the root lies b/w (0.257, 0)

$a_2 = 0.257 \quad b_2 = 0$

$f(a_2) = -0.011 \quad f(b_2) = 1$

$$a_3 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{0.257(1) - 0(-0.257)}{1 + 0.011}$$

$$= \frac{0.257}{1.011} = 0.2542$$

$f(a_3) = f(0.254) = (0.254)^3 - 4(0.254) + 1$

$= 0.00033$

then the root lies b/w $(0.254, 0.257)$

$a_3 = 0.254 \quad b_3 = 0.257$

$f(a_3) = 0.00033 \quad f(b_3) = -0.011$

$$a_4 = \frac{a_3 f(b_3) - b_3 f(a_3)}{f(b_3) - f(a_3)}$$

$$= \frac{0.254(-0.011) - 0.257(0.00033)}{-0.011 - 0.00033}$$

$$= \frac{-0.0002 - 0.0000007}{-0.01103}$$

2) $x^3 - x - 4 = 0$

$f(x) = x^3 - x - 4$

$f(0) = (0)^3 - 0 - 4 = -4\ (<0)$

$f(1) = (1)^3 - 1 - 4 = -4\ (<0)$

$f(2) = (2)^3 - 2 - 4 = 2\ (>0)$

then the roots lies b/w [1, 2]

$a_0 = 1 \qquad b_0 = 2$

$f(a_0) = -4 \qquad f(b_0) = 2$

By using regular - falsi method

$$a_1 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f_0(b_0) - f(a_0)}$$

$$= \frac{1(2) - 2(-4)}{2+4}$$

$$= \frac{2+8}{6} = \frac{10}{6} = \frac{5}{3} = 1.666$$

$f(a_1) = f(1.666) = (1.666)^3 - (1.666) - 4$

$= -1.041$

then the roots lies b/w [1.666, 2]

$$a_2 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)}$$

$$= \frac{1.666(2) - 2(-1.041)}{2+1.041}$$

$$= \frac{3.332 + 2.082}{3.041} = 1.780$$

$f(a_2) = f(1.780) = (1.780)^3 - 1.780 - 4$

$= -0.140$

then the root lies b/w $[1.780, 2)$

$$a_3 = \frac{a_2 f(b_2) - b_2 f(a_2)}{f(b_2) - f(a_2)}$$

$$= \frac{1.780(2) - 2(-0.140)}{2 - (-0.140)}$$

$$= \frac{3.56 + 0.28}{2.14} = 1.794$$

* $x^3 - x - 1 = 0$

$f(x) = x^3 - x - 1$

$f(0) = (0)^3 - 0 - 1 = -1 \ (<0)$

$f(1) = (1)^3 - 1 - 1 = -1 \ (<0)$

$f(2) = (2)^3 - 2 - 1 = 5 \ (>0)$

then the roots lies b/w $(1, 2)$

$a_0 = 1 \qquad b_0 = 2$

$f(a_0) = -1 \qquad f(b_0) = 5$

By using Regular-Flasi method

$$a_1 = \frac{a_0 f(b_0) - b_0 f(a_0)}{f(b_0) - f(a_0)}$$

$$= \frac{1(5) - 2(-1)}{5-(-1)}$$

$$= \frac{5+3}{6} = \frac{8}{6} = 1.333$$

$f(a_1) = f(1.333) = (1.333)^3 - (1.333) - 1 = 0.035$

then roots $(1.333, 1)$

$$a_2 = \frac{a_1 f(b_1) - b_1 f(a_1)}{f(b_1) - f(a_1)} = \frac{1.333(-1) - 1(0.035)}{-1 - 0.035}$$

$$\frac{-1.333 - 0.035}{-1.035}$$

$$= \frac{-1.368}{-1.035}$$

$$= 1.321$$

Iteration method:-

consider an equation $f(x)=0$ which can takes in the form

$x=\phi(x)$ as $\phi(x)<1$

$\phi(x)$ is convergent

$x_1=\phi(x_0)$

$x_2=\phi(x_1)$

$x_3=\phi(x_2)$

1) Solve x^3-x-11 by using iteration method

let $f(x)=x^3-x-11$

$f(0)=0^3-0-11=-11\ (<0)$

$f(1)=1^3-1-11=-11\ (<0)$

$f(2)=2^3-2-11=-5\ (<0)$

$f(3)=(3)^3-3-11=13\ (>0)$

then the root lies b/w $[2,3]$

$$x_0=\frac{a+b}{2}=\frac{2+3}{2}=\frac{5}{2}=2.5$$

$x^3=x+11$

$x=(x+11)^{1/3}$

$\phi(x)=x=(x+11)^{1/3}$

$\phi(x_0) = (x+11)^{1/3}$

$\phi(2.5) = (2.5+11)^{1/3}$

$\phi(x_0) = 2.379$

$\phi(x_1) = (x_0+11)^{1/3}$

$= (2.379+11)^{1/3}$

$= 2.371$

$\phi(x_2) = (x_1+11)^{1/3}$

$= (2.371+11)^{1/3}$

$= 2.371$

2, $x^3 - 5x + 3$

let $f(x) = x^3 - 5x + 3$

$f(1) = 1^3 - 5(1) + 3 = -1 \ (<0)$

$f(2) = 2^3 - 5(2) + 3 = 1 \ (>0)$

then the root lie b/w $[1, 2]$

$x_0 = \frac{a+b}{2} = \frac{1+2}{2} = \frac{3}{2} = 1.5$

from the eqn $x^3 - 5x + 3 = 0$

$x^3 = 5x - 3$

$\phi(x) = x = (5x-3)^{1/3}$

$\phi(x) = (5x-3)^{1/3}$

$= (5(1.5)-3)^{1/3}$

$\phi x_0 = 1.650$

$\phi x_1 = (5x_0-3)^{1/3}$

$= (5(1.650-3)^{1/3}$

$= 1.737$

$\phi(x_2) = (5(1.737)-3)^{1/3}$

$= 1.783$

$\phi(x_3) = (5(1.783)-3)^{1/3}$

$= 1.807$

$\phi(x_4) = (5(1.807)-3)^{1/3}$

$= 1.819$

$\phi(x_5) = (5(1.819)-3)^{1/3}$

$= 1.825$

$\phi(x_6) = (5(1.825)-3)^{1/3}$

$= 1.828$

$\phi(x_7) = (5(1.828-3)^{1/3}$

$= 1830$

$\phi(x_8) = (5(1.830)-3)^{1/3}$

$= 1.831$

$$\phi(x_4) = (5(1.831) - 3)^{1/3}$$
$$= 1.831$$

H.W

3, $5x^3 - 20x + 3$

$f(x) = 5x^3 - 20x + 3$

$f(0) = 5(0) - 20(0) + 3 = 3$

$f(1) = 5(1) - 20(1) + 3 = -12 \; (<0)$

$f(2) = 5(2)^3 - 20(2) + 3 = 3.$

$f(3) = 5(3)^3 - 20(3) + 3 = 78$

$135 - 60 + 3$

$= 5(4)^3 - 0$

Newton - Raphson method:-

* let f(x) be the given operation for which we root to $f(x_0)$ is ~~real~~ real root.
* Randomly Select two values a and b Such that f(a), f(b) have opposite signs
* let the initial approximation a and b or $\frac{a+b}{2}$
* Iterative formula to find approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

∴ using newton's - Raphson method

$x^3 - 3x + 3 = 0$

Sol :- let $f(x) = x^3 - 3x - 5 = 0$

$f'(x) = 3x^2 - 3$

$f(0) = (0)^3 - 3(0) - 5 = -5 \ (<0)$

$f(1) = (1)^3 - 3(1) - 5 = -7 \ (<0)$

$f(2) = (2)^3 - 3(2) - 5 = -3 \ (<0)$

$f(3) = (3)^3 - 3(3) - 5 = 13 \ (>0)$

then the root lies b/w [2, 3]

$x_0 = \frac{a+b}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$

$f(x_0) = (2.5)^3 - 3(2.5) - 5 = 3.125$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$n = 0$

$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f'(x_0) = 3x^2 - 3$

$= 3(2.5)^2 - 3$

$= 15.75$

$x_1 = 2.5 - \frac{3.125}{15.75} = 2.5 - 0.198$

$= 2.302$

$f(x_1) = x^3 - 3x - 5 =$

$f(2.302) = (2.302)^3 - 3(2.302) - 5 = 0.292$

$f'(x_1) = f'(2.302) = 3(2.302)^2 - 3$

$= 12.897$

put $n = 1$

$$x_{1+1} = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.302 - \frac{0.292}{12.897}$$

$= 2.302 - 0.0226$

$= 2.280.$

$f(x_2) = (2.280)^3 - 3(2.280) - 5 = 0.012$

$f'(x_2) = 3(2.280)^2 - 3 = 12.594$

put $n = 2$

$$x_{2+1} = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.280 - \frac{0.012}{(12.594)}$$

$= 2.280 + 0.0009$

$= \cancel{2.284} = 2.279$

$x^2 - 5x + 3$

Let $f(x) = x^2 - 5x + 3$ $\quad f'(x) = 2x - 5$

$f(0) = (0)^2 - 5(0) + 3 = 3 \ (>0)$

$f(1) = (1)^2 - 5(1) + 3 = -1 \ (<0)$

then the roots is $(1, 0)$

$$x_0 = \frac{a+b}{2} = \frac{1+0}{2} = 0.5$$

$$f(x_0) = (0.5)^2 - 5(0.5) + 3 = 0.75$$

$$f'(x_0) = 2(0.5) - 5 = -4$$

put $n = 0$

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.5 - \frac{0.75}{-4}$$

$$= 0.5 + 0.187$$

$$x_1 = 0.687$$

$$f(x_1) = f(0.687) = (0.687)^2 - 5(0.687) + 3$$

$$= 0.036$$

$$f'(x_1) = 2(0.687)^2 - 5$$

$$= -3.626$$

$7 - \cos 7$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.687 - \frac{0.036}{-3.626}$$

$$= 0.687 + 0.009$$

$$x_2 = 0.696$$

* $x - \cos x = 0$

$f(x) = x - \cos x = 0$ — $f'(x) = 1 + \sin x$

$f(0) = 0 - \cos(0) = -1 \ (<0)$

$f(1) = 1 - \cos(1) =$

$1 - 0.999 = 0.001 \ (>0)$

then the roots $(0, 1)$

$$x_0 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5$$

$f(x_0) = (0.5) - \cos(0.5) =$

$= (0.5) - 0.999 = -0.499$

$f'(x_0) = 1 + \sin x$

$= 1 + \sin(0.5)$

$= 1 + 0.008 = 1.008$

$$x_{n+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\frac{(-0.499)}{1.008}$$

$$\frac{0.499}{1.008}$$

$$0.5 + 0.495 = 0.995$$

$$f(x_1) = (0.995) - \cos(0.995)$$

$$= (0.995) - 0.999$$

$$= -0.004$$

$$f'(x_1) = 1 + \sin(0.995)$$

$$= 1 + 0.0173 = 1.0173$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 0.995 - \left[\frac{-0.004}{1.0173}\right]$$

$$= 0.995 + 0.003 = 0.998$$

Interpolation:- Interpolation is the process of finding out of the unknown values which lies in the given set of tabulated value.

x	x_0	x_1	....	x_n
y	y_0	y_1	....	y_n

where 'x' is Independent value and 'y' is dependent value.

Netwon's forward formula for interpolation

* Let $x_0, x_1, \ldots, x_{n-1}, x_n$ a set of equidistent values of the variable x

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 \ldots\ldots x_n - x_{n-1} = h$$

Let $u = \frac{x - x_0}{h}$

The newton's forword difference formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

$$\cdots\cdots \frac{u(u-1)(u-2)\cdots\cdots(u-n+1)}{n!}\Delta^n y_0$$

1) use Newton's forward interpolation formula and given table of value of $f(x)$ when $x=4$, given

x	3	5	7	9
$y=f(x)$	180	150	120	90

Ans:- let $h = x_1 - x_0 = 5-3 = 2$.

$$u = \frac{x - x_0}{h} = \frac{4-3}{2} = \frac{1}{2} = 0.5$$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
3	180 (y_0)	$y_1 - y_0 = -30$	0	0
5	150 (y_1)	$y_2 - y_1 = -30$		
7	120 (y_2)	$y_3 - y_2$ -30		
9	90 (y_3)			

using Newton's forword for interpolation formula

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0]$$

$$= 180 + \frac{0.5}{1}(-30) + \frac{0.5(0.5-1)}{2}(0) + 0$$

$$= 180 - 15$$

$$= 165$$

* find the newtons forward interpolation formula the value of $f(2.5)$ from the following table

x	2	3	4	5
$y = f(x)$	14.5	16.3	17.5	18

∴ let $h = x_1 - x_0 = 3 - 2 = 1$

$$u = \frac{x - x_0}{h} = \frac{2.5 - 2}{1} = 0.5$$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
2	14.5	16.3-14.5 = 1.8	-0.6	-0.1
3	16.3			
		17.5-16.3 = 1.2	-0.7	
4	17.5			
5	18	18-17.5 = 0.5		

using Newton's forward for interpolation formula

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= 14.5 + \frac{0.5}{1} 1.8 + \frac{0.5(0.5-1)}{2}(-0.6) + \frac{0.5(0.5-1)(0.5-2)}{6}(-0.1)$$

$$= 14.5 + 0.9 + \left(\frac{0.5(-0.5)}{2}\right) \times (-0.6) +$$

$$= 14.5 + 0.9 + [-0.125 \times (-0.6)] + [0.0625 \times (-0.1)]$$

$$= 15.4 + 0.075 - 0.006 = 15.469$$

Newton's backward interpolation formula.

Let $x_0, x_1, x_2 \ldots\ldots x_{n-1}, x_n$ be a set of equidistant value argument x and $y_0, y_1, y_2 \ldots\ldots y_{n-1}, y_n$ be the interpolating the value of the function $y = f(x)$

$$h = x_1 - x_0$$

let $u = \frac{x - x_n}{h}$

Newton's backward interpolation formula is

$$y(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots\cdots$$

1. use Newton's backward interpolation formula find the following table of values of $f(0.63)$

x	0.30	0.40	0.50	0.60	0.70
y	0.6179	0.6554	0.6415	0.7257	0.7580

Sol:- let $x_1 - x_0 = 0.40 - 0.30 = 0.10$

$$u = \frac{x - x_n}{h} = \frac{0.63 - 0.70}{0.10}$$

$$u = \frac{-0.07}{0.10} = -0.7$$

Difference table

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
0.30	0.6179	$y_1 - y_0 = 0.0375$			
0.40	0.6554				
0.50	0.6915	$y_2 - y_2 = 0.0361$	-0.0014		
0.60	0.7257	$y_3 - y_2 = 0.0342$	-0.0019	-0.0005	
0.70	0.7580	$y_4 - y_3 = 0.0323$	-0.0019	0	0.0005

using Newton's backward interpolation formula

$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$= 0.7580 + \frac{-0.7}{1}(0.0323) + \frac{-0.7(-0.7+1)}{2}(-0.0019) + 0 + \frac{-0.7(-0.7+1)(-0.7+2)(-0.7+3)}{24} \times (0.0005)$$

$= +0.7586 - 0.02261 - 0.105(-0.0019) -$
$0.0261(0.0005)$

$= 0.7586 - 0.02261 + 0.00019 - 0.000013$
$= 0.736567$

2, find the solution using Newton's backward interpolation at $x = 1925$

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

Sol: let $x_1 - x_0 = 1901 - 1891 = 10 = h$

$$u = \frac{x - x_n}{h} = \frac{1925 - 1931}{10} = -0.6$$

Difference table:-

x	y	∇y_n	$\nabla^2 y_n$	$\nabla^3 y_n$	$\nabla^4 y_n$
1891	46	$y_1 - y_0$			
		20	-5		
1901	66	$y_2 - y_1$		2	-3
		15	-3		
				-1	
1911	81				

1921	93	$y_3 - y_2$	
		12	-4
1931	101	$y_4 - y_3$	
		8	

Using Newton's backward interpolation formula:

$$y = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$y = 101 + (-0.6)\cdot 8 + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(0.6+1)(-0.6+2)}{6}(-1) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24} \times (-3)$$

$$y = 101 + (-4.8) - 0.12(-4) + 0.056 + 0.1008$$

$$y = 101 + (-4.8) + 0.48 + 0.056 + 0.1008$$

$$= 96.83.$$

Newton's General interpolation formula (unequal intervals):-

1. Newton's divided difference formula
2. Lagrange's formula for unequal intervals

1. NEWTON'S DEVIDED DIFFERENCE FORMULA

x	x_0	x_1	x_2	x_3 .....
$y=f(x)$	y_0	y_1	y_2	y_3 ...

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0 x_1,x_2,x_3,x_4) + \ldots$$

first devided difference:-

$$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} \text{ (or) } \frac{y_1-y_0}{x_1-x_0}$$

$$f(x_1,x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \text{ (or) } \frac{y_2-y_1}{x_2-x_1}$$

Second devided difference

$$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$$

Third devided difference:-

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

1, using newton's divided difference formula from the following table of values of $f(x)$ compute $f(6)$

x	1	2	7	8
y	1	5	5	4

$x_0 = 1 \quad x_1 = 2 \quad x_2 = 7 \quad x_3 = 8$

$y_0 = 1 \quad y_1 = 5 \quad y_2 = 5 \quad y_3 = 4$

$\Delta f(x)$

$$f(x_0, x_1) = \frac{y_1 - y_0}{x_1 - x_0} = \frac{5-1}{2-1} = \frac{4}{1} = 4$$

$$f(x_1, x_2) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-5}{7-2} = \frac{0}{5} = 0$$

$$f(x_2, x_3) = \frac{y_3 - y_2}{x_3 - x_2} = \frac{4-5}{8-7} = \frac{-1}{1} = -1$$

$\underline{\Delta^2 f(x)}$

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{0-4}{7-1} = \frac{-4}{6} = \frac{-2}{3}$$

$$f(x_1, x_2, x_3) = \frac{f(x_3, x_2) - f(x_1, x_2)}{x_3 - x_1}$$

$$= \frac{-1-0}{8-2} = \frac{-1}{6}$$

$\underline{\Delta^3 F(x)}$

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2 x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

$$= \frac{\frac{-1}{6} + \frac{4}{6}}{8-1}$$

$$= \frac{\frac{-1+4}{6}}{7} = \frac{\frac{3}{6}}{7}$$

$$= \frac{1}{14}$$

By using Newton's difference interpolation formula (unequal intervals)

$$y = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3)$$

$$= 1 + (6-1)(4) + (6-1)(6-2)\left(\frac{-4}{6}\right) + (6-1)(6-2)(6-7)\left(\frac{1}{14}\right)$$

$$y = 1 + 20 - 20(0.666) + (-20)\left(\frac{1}{14}\right)$$

$$\Rightarrow 1 + 20 - 20(0.666) - 20(0.071)$$

$$= 1 + 20 - 13.32 - 1.402 = 6.278$$

Lagrange's divided difference formula :-

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} +$$

① using Lagrange's divided difference formula from the following table of value of $x = 10$

x	5	6	9	11
$y = f(x)$	12	13	14	16

Given $x = 10$

By using Lagrange's difference formula.

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= 12\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} + 13\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}$$

$$+ 14\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} + 16\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}$$

$$\Rightarrow 12\frac{(4)(1)(-1)}{(-1)(-4)(-6)} + 13\frac{(5)(1)(-1)}{(1)(-3)(-5)} + 14\frac{(5)(4)(-1)}{(4)(3)(-2)}$$

$$+16\frac{(5)(4)(1)}{(6)(5)(2)}$$

$$= 12\frac{-4}{-24} + 13\frac{-5}{+15} + 14\frac{-20}{-24} + 16\frac{20}{60}$$

$$= 12\left[\frac{4}{24}\right] + 13\left[\frac{-5}{15}\right] + 14\left[\frac{20}{24}\right] + 16\left[\frac{20}{60}\right]$$

$$= 12\left[\frac{1}{6}\right] + 13\left[-\frac{1}{3}\right] + 14\left[\frac{5}{6}\right] + 16\left[\frac{1}{3}\right]$$

$$= 1.91 - 4.329 + 11.662 + 5.328$$

$$= 14.571.$$

(2) $x = 301$

x	300	304	305	307
$y = f(x)$	2.4771	2.4829	2.4843	2.4871

Given $x = 301$

By using lagrange's difference formula

$$f(x) = f(x_0)\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + f(x_1)\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + f(x_2)\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$+ f(x_3)\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= 2.4771\left[\frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}\right]$$

$$+ 2.4829\left[\frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}\right]$$

$$+ 2.4843\left[\frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)}\right]$$

$$+ 2.4871\left[\frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)}\right]$$

$$= 2.4771\left[\frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}\right] + 2.4829\left[\frac{(1)(-4)(-6)}{(4)(-1)(-3)}\right]$$

$$+ 2.4843\left[\frac{(1)(-3)(-6)}{(5)(+1)(-2)}\right] + 2.4871\left[\frac{(1)(-3)(-4)}{(7)(3)(2)}\right]$$

$$= 2.4771\left[\frac{-72}{-2.85}\right] + 2.4829\left[\frac{24}{12}\right] + 2.4843$$

$$\left[\frac{18}{-10}\right] + 2.4871\left(\frac{12}{42}\right)$$

$$= 2.4771(0.5142) + 2.4829(2) + 2.4843(-1.8)$$

$$+ 2.4871(0.2857)$$

$$= 1.2737 + 4.9658 - 4.4717 + 0.7105$$

$$= 6.95 - 4.717$$

$$= 2.478$$

$$= 2.4771\left[\frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}\right]$$

$$+ 2.4829\left[\frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}\right]$$

$$+ 2.4843\left[\frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)}\right]$$

$$+ 2.4871\left[\frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)}\right]$$

$$= 2.4771\left[\frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}\right] + 2.4829\left[\frac{(1)(-4)(-6)}{(4)(-1)(-3)}\right]$$

$$+ 2.4843\left[\frac{(1)(-3)(-6)}{(5)(+1)(-2)}\right] + 2.4871\left[\frac{(1)(-3)(-4)}{(7)(3)(2)}\right]$$

$$= 2.4771\left[\frac{-72}{-2.85}\right] + 2.4829\left[\frac{24}{12}\right] + 2.4843$$

$$\left[\frac{18}{-10}\right] + 2.4871\ \frac{(12)}{(42)}$$

$$= 2.4771(0.5142) + 2.4829(2) + 2.4843(-1.8)$$

$$+ 2.4871(0.2857)$$

$$= 1.2737 + 4.9658 - 4.4717 + 0.7105$$

$$= 6.95 - 4.717$$

$$= 2.478$$

Unit-2

unit - II

Numerical integration, Solutions of ordinary differenitial equations with initial conditions.

Numerical integration:- The process of evaluating a definite integral from a set of tabulated values of the integrated $f(x)$ is called Numerical integration.

Tropezoidal rule:- Let $f(x)$ be a continuous function on the interval $[a,b)$ Now divide the intervals (a,b) into n equal sub intervals with each of width $h = \frac{b-a}{n}$

formula for trapezoidal rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_n\right] + 2\left[y_1 + y_2 + \cdots y_{n-1}\right]$$

1, Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ by using Tropezoidal rule by taking number of Intervals $n=4$

Given $\int_0^1 \frac{1}{1+x} dx$

comparing $\int_a^b f(x) dx$

$a=0 \quad b=1 \quad f(x) = \frac{1}{1+x}, \; n=4$

where $h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

	a	a+h		
x	0	0.25	0.5	0.75
$y=f(x)$	1	$\frac{1}{1+0.25} = 0.8$	$\frac{1}{1+0.5} = 0.66$	0.571

1
0.5

By using Trapozaidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_4) + 2(y_1+y_2 \cdots y_{n-1})\right]$$

$$= \int_0^1 \frac{1}{1+x} dx = \frac{0.25}{2}\left[(1+0.5) + 2(0.8+0.66+0.571)\right]$$

$$= 0.125\left[1.5 + 2(2.031)\right]$$

$$= 0.125\left[1.5 + 4.074\right)$$

$$= 0.125(5.574) = 0.6967$$

2. Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by using Trapezoidal rule

given $\int_0^6 \frac{1}{1+x^2} dx$ comparing $\int_a^b f(x) dx$

$a=0$ $b=6$ $f(x) = \frac{1}{1+x^2}$, $n=6$

where $h = \frac{b-a}{n} = \frac{6-0}{6} = 1$

x	0	1	2	3	4	5	6
$y=f(x)$	1	0.5	0.2	0.1	0.058	0.038	0.027

$\frac{1}{1+x^2} = \frac{1}{(1+1)} = 0.5$

$\frac{1}{1+x_i^2} = \frac{1}{(1+(2)^2)} = \frac{1}{5} = 0.2$

by using Trapozaidal rule

$\int_a^b f(x) dx = \frac{h}{2} [y_0 + y_6] + 2(y_1 + y_2 + \dots y_{n-1})$

$= \int_0^6 \frac{1}{1+x^2} dx = \frac{1}{2} [(1+0.027) + 2(0.5+0.2+0.1+0.058+0.038)$

$= 0.5 [1.027 + 2(0.896)$

$= 0.5 [1.027 + 1.792]$

$= 0.5 [2.819]$

$= 1.4095$

3, Evaluate $\int_0^1 x^3\,dx$ with 5 sub-intervals

By using Trapezoidal rule.

Sol:- given $\int_0^1 x^3\,dx$

comparing $\int_a^b f x\,dx$

$a = 0 \quad b = 1 \quad f(x) = x^3 \quad n = 5$

where $h = \frac{1-0}{5} = 0.2$

x	0	0.2	0.4	0.6	0.8	1
$f(x)$	0	0.008	0.064	0.216	0.512	1

By using Traposidal rule

$\int_a^b f(x)\,dx = \frac{0.2}{2}[0+1] + 2[0.008 + 0.064 + 0.216 + 0.512]$

$= 0.1[0+1] + 2[0.8]$

$= 0.1[1] + [1.6]$

$0.1[2.6]$

$= 0.26$

4, Evaluate $\int_0^5 \sin x \, dx$ with 5 sub-inter

By using Trapezoidal rule

given $\int_0^5 \sin x \, dx$

$a = 0$ $b = 5$ $f(x) = \sin x$ $n = 5$

where $h = \frac{5-0}{5} = 1$

x	0	1	2	3	4	5
$f(x)$	0	0.017	0.034	0.052	0.069	0.087

$= \frac{1}{2}[0.087 + 0] + 2[0.017 + 0.034 + 0.052 + 0.069]$

$= 0.5[0.087] + 2[0.172]$

$= 0.5[0.087] + [0.344$

$= 0.5[0.431] = 0.2155$

Simpson's $\frac{1}{3}$ rule :-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5\ldots\ldots)+2(y_2+y_4+y_6\ldots\ldots+y_n)\right]$$ where

$$h = \frac{b-a}{n}$$

1. Obtain the value of $\int_0^1 \frac{1}{1+x^2}\,dx$ using Simpson's $\frac{1}{3}$rd rule by dividing the interval at [0,1] into 4 equal parts.

given $\int_0^1 \frac{1}{1+x^2}\,dx$

comparing $\int_a^b f(x)\,dx$

$a = 0$, $b = 1$, $f(x) = \frac{1}{1+x^2}$, $n = 4$

$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$

x	0	0.25	0.5	0.75	1
$y = f(x)$	$\frac{1}{1+0} = 1$	0.941	0.8	0.641	0.5

By using Simpson's $\frac{1}{3}$ rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{3}\left[[1+0.5] + 4(0.9412 + 0.641) + 2(0.8)\right]$$

$$= 0.083[1.5] + 4(1.5812) + 2(0.8)$$

$$= 0.1245 + 6.3248 + 1.6 =$$

$$= 0.083[1.5 + 6.3248 + 1.6]$$

$0.083 (9.428)$

$= 0.7858$

(2) Evaluat. $\int^2 e^x$

Simpson's $\frac{3}{8}$ rule:-

$$\int_a^b f(x)\,dx = \frac{3}{8}h\,[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots) + 2(y_3+y_6+y_9+\dots)]$$

where $h = \frac{b-a}{n}$

(1) Evaluate $\int_0^6 \frac{1}{1+x^2}\,dx$ by Simpson's $\frac{3}{8}$th rule

Sol:- Given $\int_0^6 \frac{1}{1+x^2}\,dx$

$a=0$, $b=6$, $f(x)=\frac{1}{1+x^2}$, $n=6$

where $h = \frac{b-a}{n} = \frac{6-0}{6} = \frac{6}{6} = 1$

x	0	1	2	3	4	5	6
$y=f(x)$	1	0.5	0.2	0.1	0.058	0.038	0.027

By using Simpson's $\frac{3}{8}$ rule

$$\int_0^6 \frac{1}{1+x^2}\,dx = \frac{3(1)}{8}[(1+0.027)+3(0.5+0.2+0.058+0.038]+2[0.1]$$

$= 0.375[1.027+2.388+0.2]$

$= 1.355$

Taylor's Series:-

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots$$

① Evaluate use taylor's series method to find y at $x = 0.1, 0.2, 0.3$ considering terms upto the third degree given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

Sol:- given $\frac{dy}{dx} = x^2 + y^2$ and $y(0) = 1$

$x_0 = 0 \quad y_0 = 1 \quad f(x_0) = y_0$

By using Taylor's Series

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \ldots$$

$$\Rightarrow y(x) = y(0) + \frac{x-0}{1!} y'(0) + \frac{(x-0)^2}{2!} y''(0) + \frac{(x-0)^3}{3!} y'''(0)$$

$= y(0) + x\, y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0)$

dit $y' = x^2 + y^2$

$y'(0) = x_0^2 + y_0^2$

$= (0)^2 + (1)^2$

$= 1$

$y'' = 2x + 2y \cdot y'$

$y''(0) = 2(x_0) + 2y_0 y_0'$

$= 2(0) + 2(1)(1)$

$= 0 + 2 = 2$

$y''' = 2 + 2[y \cdot y'' + y'^2]$ $\quad y \cdot y'$, $u\,v$, $uv' + vu'$

$= 2 + 2[y_0 y'' + y_0'^2]$

$\Rightarrow 2 + 2[1(2) + (1)^2]$

$\Rightarrow 2 + 2[2 + 1]$

$\Rightarrow 2 + 2[3]$

$\Rightarrow 2 + 6 = 8$

$y(x) = y(0) + x\, y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0)$

$y(x) = 1 + x + x^2 + \frac{4x^3}{3} \longrightarrow (1)$

When $x = 0.1, 0.2, 0.3$ 1.426.

$x = 0.1$ in substitute eq^n (1)

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{4(0.1)^3}{3}$$

$= 1.1113$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{4(0.2)^3}{3}$$

$= 1.2506$

$$y(0.3) = 1 + 0.3 + (0.3)^2 + \frac{4(0.3)^3}{3}$$

$= 1.426$

2) using taylor's series method find the approximate value at $x = 0.2$ for the ordinary differential equation at $y' - 2y = 3e^x$, $y(0) = 0$

Sol:- Given $y' - 2y = 3e^x$ $y(0) = 0$

$f(x_0) = y_0$

$$y' = 3e^x + 2y$$

$$\frac{dy}{dx} = 3e^x + 2y$$

$x_0 = 0$ $y_0 = 0$

0.01

0.003 = 0.012 = 0.0004

By using taylor's series method-

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots$$

$$y(x) = y(0) + x\, y'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \cdots$$

Take $y' = 3e^x + 2y$

$$y'(0) = 3e^x + 2(y_0)$$
$$= 3e^x + 2(0)$$
$$= 3e^x = 3$$

$$y'' = 3e^{x_0} + 2y_0'$$
$$= 3 + 2(3) = 9$$

$$y'''(0) = 3 + 2\, y_0''$$
$$= 3 + 2(9) = 21$$

$$y'''' = 3e^x + 2y'''$$
$$= 3e^{x_0} + 2\cdot y_0'''$$
$$= 3 + 2(21)$$
$$= 3 + 42$$
$$= 45$$

$$y(x) = 0 + x.3 + \frac{x^2}{2}9 + \frac{x^3}{6}21 + \frac{x^4}{24}45 \quad \text{--- (1)}$$

$= \sout{3x+}$

$$y(0.2) + 3(0.2) + \frac{(0.2)^2}{2}\times 9 + \frac{(0.2)^3}{6}(21) + \frac{(0.2)^4}{24}(45) + \dots$$

$$= 0.6 + \frac{9(0.04)}{2} + \frac{21(0.008)}{6} + \frac{45(0.0016)}{24}$$

$$= 0.6 + 0.18 + 0.028 + 0.003 + \dots$$

$$= 0.811$$

Picard's method :-

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\, dx$$

(1) $y' = y - x^2$ and $y(0) = 1$ do upto 4th approximations and find the value of $y(0.1)$ and $y(0.2)$ by Picard's method.

Sol Given : $y' = y - x^2 = f(x,y)$, $f(x_0) = y_0$

$x_0 = 0$, $y_0 = 1$.

By using Picard's method

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})\, dx$$

put $n=1$

$$y^{(1)} = 1 + \int_0^x f(x, y^{(1-1)})\, dx$$

$$= 1 + \int_0^x f(x, y_0)\, dx$$

$$= 1 + \int_0^x f(x, 1)\, dx \quad (y - x^2)$$

$$= 1 + \int_0^x (1 + x^2)\, dx$$

$$= 1 + [x]_0^x - \left[\frac{x^3}{3}\right]_0^x$$

$$y^{(1)} = 1 + [x - 0] - \left[\frac{x^3}{3} - 0\right]$$

$$y^{(1)} = 1 + x - \frac{x^3}{3}$$

put $n=2$

$$y^{(2)} = 1 + \int_0^x f(x, y^{2-1})\, dx$$

$$= 1 + \int_0^x f(x, y^{(1)})\, dx$$

$$= 1 + \int_0^x f\left(x, 1 + x - \frac{x^3}{3}\right) dx$$

$$y^{(2)} = 1 + \int_0^x \left(1 + x - \frac{x^3}{3} - x^2\right) dx$$

$$= 1 + [x]_0^x + \left[\frac{x^2}{2}\right]_0^x - \frac{1}{3}\left[\frac{x^4}{4}\right]_0^4 - \left[\frac{x^3}{3}\right]_0^x$$

$$= 1 + [x - 0] + \left[\frac{x^2}{2} - 0\right] - \frac{1}{3}\left[\frac{x^4}{4} - 0\right] - \left[\frac{x^3}{3} - 0\right]$$

$$y^2 = 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3}$$

put $n = 3$

$$y^3 = 1 + \int_0^x f(x, y^{3-1})\, dx$$

$$= 1 + \int_0^x f(x, y^{(2)})\, dx$$

$$= 1 + \int_0^x f\left(x, \underline{1 + x + \frac{x^2}{2} - \frac{x^4}{12}}\right)$$

$$y^{(3)} = 1 + \int_0^x 1 + x + \frac{x^2}{2} - \frac{x^4}{12} - \frac{x^3}{3} - x^2 \Big]$$

$$y^{(3)} = 1 + [x]_0^x + \left[\frac{x^2}{2}\right]_0^x + \frac{1}{2}\left[\frac{x^3}{3}\right]_0^x - \frac{1}{12}\left[\frac{x^5}{5}\right] - \frac{1}{3}\frac{x^4}{4} - \frac{x^3}{3}$$

$$y^{(3)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3}$$

put $n = 4$

$$y^{(4)} = 1 + \int_0^x f(x, y^{4-1})\,dx$$

$$= 1 + \int_0^x f(x, y^3)\,dx$$

$$= 1 + \int_0^x f\left(x, 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3}\right)$$

$$y^{(4)} = 1 + \int_0^x 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3} - x^2$$

$$= 1 + [x]_0^x + \left[\frac{x^2}{2}\right]_0^x + \frac{1}{2}\left[\frac{x^3}{3}\right]_0^x + \frac{1}{6}\left[\frac{x^4}{4}\right] - \frac{1}{60}\left[\frac{x^6}{6}\right]$$

$$- \frac{1}{12}\left[\frac{x^5}{5}\right]_0^x - \frac{1}{3}\left[\frac{x^4}{4}\right]_0^x - \left[\frac{x^3}{3}\right]_0^x$$

$$y^{(4)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^6}{360} - \frac{x^5}{60} - \frac{x^4}{12} - \frac{x^3}{3}$$

$x = 0.1$

$$y^{(4)} = 1 + (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24} - \frac{(0.1)^6}{360} - \frac{(0.1)^5}{60} - \frac{(0.1)^4}{12} - \frac{(0.1)^3}{3}$$

$$= 1.1 + 0.005 + \frac{0.003}{6} + \frac{0.0001}{24} - \frac{0.00001}{360}$$

$$- \frac{0.00001}{60} - \frac{0.0001}{12} - \frac{0.001}{3}$$

$$= 1 + 0.1 - 0.005 - 0.00016 - 0.0000041 -$$

$$0.0000000027 - 0.00000016 - 0.0000083 - 0.00033$$

$$= 1 + 0.1 + 0.0055$$

c) given $= 1.1055$

$$y^{(4)} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^6}{360} - \frac{x^5}{60} -$$

$$\frac{x^4}{12} - \frac{x^3}{3}$$

$$y^{(4)} = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} + \frac{(0.2)^4}{24} -$$

$$\frac{(0.2)^6}{360} - \frac{(0.2)^5}{60} - \frac{(0.2)^4}{12} - \frac{(0.2)^3}{3}$$

$$= 1 + 0.2 + 0.02 + 0.001 - 0.00006 -$$

$$0.0000001 - 0.000005 - 0.0001 -$$

$$0.002 = 1.216$$

d)

Given the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2+1}$$

with the initial condition $y=0$ when $x=0$. use Picard's method to obtain y for $x = 0.25, 0.5$ and 1.0 correct to three decimal places.

$$\frac{dy}{dx} = \frac{x^2}{y^2+1} = y'$$

$y_0 = 0 \quad x_0 = 0$

$n = 1$

$$y^{(1)} = 1 + \int_0^x f(x, y^{(1-1)}) \, dx$$

$$y = 1 + \int_0^x f(x, y_0) \, dx$$

$$= 1 + \int_0^x \frac{x^2}{y^2+1} \, dx$$

Euler's method:-

In euler's method if the differential equation is $\frac{dy}{dx} = f(x,y)$

Where $x = x_0$ and $y = y_0$

$x_1 = x_0 + h, \quad x_2 = x_1 + h$

then, the general solution is given by

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

1, using Euler's method solve for $y(2)$ from $\frac{dy}{dx} = 3x^2+1$, $y(1) = 2$ Taking step size. (i) $h = 0.5$ (ii) $h = 0.25$

Sol:- Given $\frac{dy}{dx} = 3x^2+1$

$f(x_0) = y_0 \Rightarrow y(1) = 2$

$x_0 = 1, \quad y_0 = 2$

$x_1 = x_0 + h = 1 + 0.5$

$x_1 = 1.5$

To find $y(2)$ by using step length $h = 0.5$ by using euler's method put $n = 0$

$y_{0+1} = y_0 + h f(x_0, y_0)$

$\Rightarrow y_1 = 2 + 0.5 f(\overset{x,y}{1,2})$

$y_1 = 2 + 0.5 (3(1)^2 + 1)$

$y_1 = 2 + 0.5(4)$

$y(x_1) = 4$

$y(1.5) = 4$

To find $y(2)$ by using step length $h = 0.5$

let $x_2 = x_1 + h$

$= 1.5 + 0.5$

$= 2.$

$y_{1+1} = y_1 + h f(x_1, y_1)$

$y(2) = y_1 + 0.5 f(3(x)^2 + 1)$

$y(2) = 4 + 0.5 (1.5, 4)$

$y(2) = 4 + (0.5)(3(1.5)^2 + 1)$

$= 4 + (0.5)(7.75)$

$= 4 + 3.875 = 7.875$

(ii) $h = 0.25$

Given $\frac{dy}{dx} = 3x^2 + 1$

$f(x_0) = y_0$; $f(1) = 2$

$x_0 = 1$; $y_0 = 2$

$x_1 = x_0 + h = 1 + 0.25 = 1.25$

by using Euler's method put $n = 0$

$y_{0+1} = y_0 + h f(x_0, y_0)$

$y_1 = 2 + 0.25 f(1, 2)$

$= 2 + 0.25 (3(x)^2 + 1)$

$y_1 = 2 + 0.25 (3(1)^2 + 1)$

$= 2 + 0.25 (4)$

$= 3$

To find $y(2)$ by using step length.

$x_2 = x_1 + h = 1.25 + 0.25$

$= 1.5$

by using Euler's method put $n = 1$

$y_{1+1} = y_1 + h f(x_1, y_1)$ 1.25, 3

$y_2 = y_1 + h f(x_1, y_1)$

$y_2 = 3 + 0.25 (3(x)^2 + 1)$

$= 3 + 0.25 (3(1.25)^2 + 1)$

$= 3 + 0.25 (5.687) = 4.421$

To find $y(2)$

$x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

by using Euler's method put, $n = 2$

$y_3 = y_2 + h\, f(x_2, y_2)$

$y_3 = 4.42 + 0.25\,(1.5,\ 4.42)$

$y_3 = 4.42 + 0.25\,(3(1.5)^2 + 1)$

$= 4.42 + 0.25\,(7.75)$

$= 4.42 + 1.9375 = 6.357$

$x_4 = x_3 + h = 1.75 + 0.25$

$= 2$

$y_4 = y_3 + h\, f(x_3, y_3)$ $(1.75,\ 6.357)$

$= 6.357 + 0.25\,(3(x)^2 + 1)$

$= 6.357 + 0.25\,(3(1.75)^2 + 1)$

$= 6.357 + 0.25\,(10.1875)$

$= 6.357 + 2.5468 = 8.90$

2, To solve $\frac{dy}{dx} = x + y$ with $y(0) = 1$, then find $y(0.2)$ using method by taking step size $h = 0.1$

$$\frac{dy}{dx} = x+y$$

$y(x_0) = y_0 = y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

$h = 0.1$

$x_1 = x_0 + h = 0 + 0.1$

$= 0.1$

put $n = 0$

$y_{0+1} = y_0 + h\, f(x_0, y_0)$

$y_1 = 1 + 0.1\, f(\overset{x,\,y}{0,1})$

$y_1 = 1 + 0.1\,(0+1)$

$y_1 = 1 + 0.1\,(1)$

$y_1 = 1 + 0.1$

$y(0.1) = 1.1$

$x_2 = x_1 + h$

$= 0.1 + 0.1 = 0.2$

put $n = 1$

$y_{1+1} = y_1 + h f(x_1, y_1)$

Runge - kutta method

$\frac{dy}{dx} = f(x, y)$, with $y(x_0) = y_0$ given then to find $y_1 = y(x_0 + h)$ we use the following Runge kutta method.

Runge kutta method (second order):-

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h, y_0 + k_1)$

Runge - kutta method (Fourth order):-

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3)$.

① $\frac{dy}{dx} = 1 + xy$, $y(0) = 2$, find $y(0.1)$, $y(0.2)$ and $y(0.3)$ by runge - kutta method.

② If $y' = x + y$ with $y(0) = 1$, then find $y(0.1)$, using runge - kutta method.

Sol:- let $f(x,y) = x+y$

$y(x_0) = y_0$

$y(0) = 1$

$x_0 = 0 \quad y_0 = 1 \quad h = x - x_0 = 0.1 - 0 = 0.1$

using Runge-kutta method

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \rightarrow (1)$

$k_1 = hf(x_0, y_0)$

$= 0.1(x_0 + y_0)$

$= 0.1(0+1)$

$= 0.1(1)$

$k_1 = 0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.1(x_0 + \frac{h}{2} + y_0 + \frac{k_1}{2})$

$= 0.1(0 + \frac{0.1}{2} + 1 + \frac{0.1}{2})$

$= 0.1(\frac{0.1 + 2 + 0.1}{2})$

$= 0.1(1.1)$

$= 0.11$

$k_3 = hf(x_0 + \frac{h}{3}, y_0 + \frac{k_2}{2})$

$= 0.1(x_0 + \frac{h}{3} + y_0 + \frac{k_2}{2})$

$= 0.1(0 + \frac{0.1}{3} + 1 + \frac{0.11}{2})$

~~$= 0.1(0.1 + 6 + 0.1$~~

$= 0.1(\frac{0.1}{3} + 1 + \frac{0.11}{2})$

$= 0.1\left(\frac{0.2+6+0.33}{6}\right)$

$= 0.1(1.08883)$

$= 0.10883$

$k_4 = hf(x_0+h, y_0+k_3)$

$= 0.1(x_0+h+y_0+k_3)$

$= 0.1(0+0.1+1+0.108)$

$= 0.1(1.208)$

$= 0.1208$

substitute k_1, k_2, k_3, k_4 in eqn ①

$y_1 = 1+\frac{1}{6}(0.1+2(0.11)+2(0.108)+0.1208)$

$y_1 = 1+\frac{1}{6}(0.1+0.22+0.216+0.1208)$

$= 1+\frac{1}{6}(0.6568)$

$= 1+\frac{0.6568}{6}$

$= 1+0.1094$

$= 1.1094$

②

(3) Apply R.K. method of 2nd order to Solve initial value problem $\frac{dy}{dx} = x+y$

$y(0) = 1$, find at $y(0.2)$ taking $h = 0.1$

Sol:- let $f(x,y) = x+y$

$y(x_0) = y_0$

$x_0 = 0 \quad y_0 = 1, \; h = 0.1$

using R.K method of 2nd order

$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$.

where $k_1 = hf(x_0, y_0)$

$= 0.1 f(0,1)$

$= 0.1(0+1)$

$= 0.1(1)$

$k_1 = 0.1$

$k_2 = hf(x_0+h, y_0+k_1)$

$= 0.1 f(0+0.1, 1+0.1)$

$= 0.1 f(0.1, 1.1)$

$= 0.1 (0.1 + 1.1)$

$= 0.12$

By using R.K method

$y_1 = 1 + \frac{1}{2}(0.1 + 0.12)$

$= 1.11$

$x_1 = x_0 + h$

$= 0 + 0.1$

$= 0.1$

$k_1 = 0.1 f(x_1, y_1)$

$= 0.1 f(0.1, 1.11)$

$= 0.1 (0.1 + 1.11)$

$= 0.121$

$k_2 = 0.1 f(x_1+h, y_1+k_1)$

$= 0.1 f(0.1+0.1, 1.11+0.121)$

$= 0.1 f(0.2, 1.231)$

$= 0.1431$

$k = \frac{1}{2}(k_1 + k_2)$

$= \frac{1}{2}(0.121 + 0.1431)$

$= 0.13205$

$y_2 = y_0 + k$

$= 1.11 + 0.13205$

$= 1.242$

$x_2 = x_1 + h = 0.1 + 0.1$

$= 0.2.$

<u>Milne's predictor and correcter method</u>:

<u>Milne's predicter formula</u>:

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y'_{n-2} - y'_{n-1} + 2y'_n\right]$$

<u>Milne's correcter formula</u>:

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1}\right]$$

(1) given that $\frac{dy}{dx} = x - y^2$ and the data $y(0)=0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$, $y(0.6) = 0.1762$. compute y at $x = 0.8$ by applying milne's method.

So:- we prepare the following table

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y'_0 = 0 - 0^2 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = 0.2 - (0.02)^2 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = 0.4 - (0.0795)^2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = 0.6 - (0.1762)^2 = 0.5689$
$x_4 = 0.8$	$y_4 = ?$	$y'_4 = ?$

we have the Predictor formula is

put $n = 3$, $h = x_1 - x_0 = 0.2 - 0 = 0.2$

~~$y_{3+1} = y_{3-2} + \frac{h}{3}[y'_{3-1} + 4y'_3 + y'_{3+1}]$~~

~~$y_4 = y_1 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$~~

$$y_{3+1} = y_{3-3} + \frac{4h}{3}[2y'_{3-2} - y'_{3-1} + 2y'_3]$$

$$y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 0 + \frac{4(0.2)}{3}[2(0.1996) - 0.1927 + 2(0.5689)]$$

$$= 0 + 0.266[0.3992 - 0.1927 + 1.1378]$$

$$= 0 + 0.266[1.3443]$$

$$= 0.3575$$

$$y_4' = x - y^2$$

$$= 0.8 - (0.3575)^2$$

$$= 0.8 - 0.12780$$

$$= 0.6722$$

Next we have corrector formula

$$y_{n+1} = y_{n-2} + \frac{h}{3}\left[y'_{n-1} + 4y'_n + y'_{n+1} \right]$$

Put $n = 3$

$$y_{3+1} = y_{3-1} + \frac{0.2}{3}\left[y'_{3-1} + 4y'_3 + y'_{3+1} \right]$$

$$y_4^{(c)} = y_2 + \frac{0.2}{3}\left[y'_2 + 4y'_3 + y'_4 \right]$$

$$= 0.0795 + 0.0666\left[0.3937 + 4(0.5689) + 0.6722\right]$$

$$= 0.0795 + 0.0666\left[3.3415\right]$$

$$= 0.0795 + 0.2225$$

$$= 0.3020$$

$$y_4' = x_4 - y_4^2$$

$$= 0.8 - (0.3020)^2$$

$$= 0.8 - 0.0912$$

$$y_4' = 0.7088$$ //

substituting the value of $y_4^{(c)}$ again in corrector formula

$$y_4^{(c)} = 0.0795 + 0.0666 [0.3937 + 4(0.5689) + 0.7088]$$

$= ~~0.3046~~$

$= 0.0795 + 0.0666 [3.3781]$

$y(0.8) = 0.3044.$ $\qquad$ $\underline{0.3046}$